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Research Article

Estimates of $\mathcal{M}$-Harmonic Conjugate Operator

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We define the $\mathcal{M}$-harmonic conjugate operator K and prove that for $1 < p < \infty$, there is a constant C_p such that $\int_S |Kf|^p \omega d\sigma \le C_p \int_S |f|^p \omega d\sigma$ for all $f \in L^p(\omega)$ if and only if the nonnegative weight ω satisfies the A_p-condition. Also, we prove that if there is a constant C_p such that $\int_S |Kf|^p v d\sigma \le C_p \int_S |f|^p w d\sigma$ for all $f \in L^p(w)$, then the pair of weights (v, w) satisfies the A_p-condition.

1. Introduction

Let B be the unit ball of $\mathbb{C}^n$ with norm $|z| = \langle z, z\rangle^{1/2}$ where $\langle , \rangle$ is the Hermitian inner product, let S be the unit sphere, and, σ be the rotation-invariant probability measure on S.

In [1], for $z \in B$, $\xi \in S$, we defined the kernel $K(z, \xi)$ by

$$iK(z,\xi) = 2C(z,\xi) - P(z,\xi) - 1, \tag{1.1}$$

where $C(z,\xi) = (1 - \langle z,\xi\rangle)^{-n}$ is the Cauchy kernel and $P(z,\xi) = (1-|z|^2)^n/|1-\langle z,\xi\rangle|^2 n$ is the invariant Poisson kernel. Thus for each $\xi \in S$, the kernel $K(,\xi)$ is $\mathcal{M}$-harmonic. And for all $f \in A(B)$, the ball algebra, such that $f(0)$ is real, the reproducing property of $2C(z,\xi) - 1$ (3.2.5 of [2]) gives

$$\int_S K(z,\xi)\,\mathrm{Re}\, f(\xi)\,d\sigma(\xi) = -i\big(f(z) - \mathrm{Re}\, f(z)\big) = \mathrm{Im}\, f(z). \tag{1.2}$$

For that reason, $K(z,\xi)$ is called the $\mathcal{M}$-harmonic conjugate kernel.

For $f \in L^1(S)$, Kf, the $\mathcal{M}$-harmonic conjugate function of f, on S is defined by

$$(Kf)(\zeta) = \lim_{r \to 1} \int_S K(r\zeta, \xi) f(\xi) d\sigma(\xi), \tag{1.3}$$

since the limit exists almost everywhere. For $n = 1$, the definition of Kf is the same as the classical harmonic conjugate function [3, 4]. Many properties of $\mathcal{M}$-harmonic conjugate function come from those of Cauchy integral and invariant Poisson integral. Indeed the following properties of Kf follow directly from Chapters 5 and 6 of [2].

(1) As an operator, K is of weak type (1.5) and bounded on $L^p(S)$ for $1 < p < \infty$.

(2) If $f \in L^1(S)$, then $Kf \in L^p(S)$ for all $0 < p < 1$ and if $f \in L \log L$, then $Kf \in L^1(S)$.

(3) If f is in the Euclidean Lipschitz space of order α for $0 < \alpha < 1$, then so is Kf.

Also, in [1], it is shown that K is bounded on the Euclidean Lipschitz space of order α for $0 < \alpha < 1/2$, and bounded on BMO.

In this paper, we focus on the weighted norm inequality for $\mathcal{M}$-harmonic conjugate functions. In the past, there have been many results on weighted norm inequalities and related subjects, for which the two books [3, 4] provide good references. Some classical results include those of M. Riesz in 1924 about the L^p boundedness of harmonic conjugate functions on the unit circle for $1 < p < \infty$ [3, Theorem 2.3 of Chapter 3] and [3, Theorems 6.1 and 6.2 of Chapter 6] about the close relation between A_p-condition of the weight and the L^p boundedness of the Hardy-Littlewood maximal operator and Hilbert transform on $\mathbb{R}$. In 1973, Hunt et al. [5] proved that, for $1 < p < \infty$, conjugate functions are bounded on weighted measured Lebesgue space if and only if the weight satisfies A_p-condition. It should be noted that in 1986 the boundedness of the Cauchy transform on the Siegel upper half-plane in $\mathbb{C}^n$ was proved by Dorronsoro [6]. Here in this paper, we provide an analogue of that of [5] and [3, Theorems 6.1 and 6.2 of Chapter 6].

To define the A_p-condition on S, we let ω be a nonnegative integrable function on S. For $p > 1$, we say that ω satisfies the A_p-condition if

$$\sup_Q \frac{1}{\sigma(Q)} \int_Q \omega d\sigma \left(\frac{1}{\sigma(Q)} \int_Q \omega^{-1/(p-1)} d\sigma \right)^{p-1} < \infty, \tag{1.4}$$

where $Q = Q(\xi, \delta) = \{\eta \in S : d(\xi, \eta) = |1 - \langle \xi, \eta \rangle|^{1/2} < \delta\}$ is a nonisotropic ball of S.

Here is the first and the main theorem.

Theorem 1.1. *Let ω be a nonnegative integrable function on S. Then for $1 < p < \infty$, there is a constant C_p such that*

$$\int_S |Kf|^p \omega d\sigma \le C_p \int_S |f|^p \omega d\sigma \quad \forall f \in L^p(\omega) \tag{1.5}$$

if and only if ω satisfies the A_p-condition.

In succession of classical weighted norm inequalities, starting from Muckenhoupt's result in 1975 [7], there have been extensive studies on two-weighted norm inequalities. Here, we define the A_p-condition for two weights. For a pair (v,w) of two nonnegative integrable functions, we say that (v,w) satisfies the A_p-condition if

$$\sup_Q \frac{1}{\sigma(Q)} \int_Q v\,d\sigma \left(\frac{1}{\sigma(Q)} \int_Q w^{-1/(p-1)} d\sigma \right)^{p-1} < \infty, \tag{1.6}$$

where Q is a nonisotropic ball of S. As mentioned above, in [7], Muckenhoupt derives a necessary and sufficient condition on two-weighted norm inequalities for the Poisson integral operator, and then in [8], Muckenhoupt and Wheeden provided two-weighted norm inequalities for the Hardy-Littlewood maximal operator and the Hilbert transform. We admit that there are, henceforth, numerous splendid results on two-weighted norm inequalities but left unmentioned here.

In this paper we provide a two-weighted norm inequality for $\mathcal{M}$-harmonic conjugate operator as our next theorem, by the method somewhat similar to the proof of the main theorem. For a pair (v,w), the generalization of the necessity in Theorem (1.5) is as follows.

Theorem 1.2. *Let (v,w) be a pair of nonnegative integrable functions on S. If for $1 < p < \infty$, there is a constant C_p such that*

$$\int_S |Kf|^p v\,d\sigma \le C_p \int_S |f|^p w\,d\sigma \quad \forall f \in L^p(w), \tag{1.7}$$

then the pair (v,w) satisfies the A_p-condition.

The proofs of Theorems 1.1 and 1.2 will be given in Section 2. We start Section 2 by introducing the sharp maximal function and a lemma on the sharp maximal function, which plays an important role in the proof of the main theorem. In the final section, as an appendix, we introduce John-Nirenberg's inequality on S and then, as an application, we attach some properties of A_p weights on S in relation with BMO, which are similar to those on the Euclidean space.

2. Proofs

Definition 2.1. For $f \in L^1(S)$ and $0 < p < \infty$, the sharp maximal function $f^{\#p}$ on S is defined by

$$f^{\#p}(\xi) = \sup_Q \left(\frac{1}{\sigma(Q)} \int_Q |f - f_Q|^p d\sigma \right)^{1/p}, \tag{2.1}$$

where the supremum is taken over all the nonisotropic balls Q containing ξ and f_Q stands for the average of f over Q.

The sharp maximal operator $f \mapsto f^{\#p}$ is an analogue of the Hardy-Littlewood maximal operator M, which satisfies $f^{\#p}(\xi) \le 2Mf(\xi)$. The proof of the following lemma is essentially the same as that of the Theorem 2.20 of [4]; so we omit its proof.

Lemma 2.2. *Let $0 < p < \infty$ and ω satisfy A_p-condition. Then there is a constant C_p such that*

$$\int_S (Mf)^p \omega d\sigma \leq C_p \int_S \left(f^{\#1}\right)^p \omega d\sigma, \tag{2.2}$$

for all $f \in L^p(\omega)$.

Now we will prove Theorem 1.1.

Proof of Theorem 1.1. First, we prove that (1.5) implies that ω satisfies the A_p-condition.
If $\xi, \eta \in S$, then by a direct calculation we get

$$K(\xi,\eta) = \frac{(1-\langle \eta,\xi\rangle)^n (2-(1-\langle \xi,\eta\rangle)^n)}{|1-\langle \xi,\eta\rangle|^{2n}}. \tag{2.3}$$

If $\xi \neq -\eta$ and $(1-\langle \eta,\xi\rangle)^n \ (2-(1-\langle \xi,\eta\rangle)^n) = 0$, then we get $\xi = \eta$. Thus if $\xi \neq \eta$, then for $\xi \approx \eta$, we have $(\operatorname{Re} K(\xi,\eta))(\operatorname{Im} K(\xi,\eta)) \neq 0$. Hence there exist positive constants δ and $\widetilde{C}$ such that

$$\left|\int_{0<d(\xi,\eta)<\delta} K(\xi,\eta) f(\eta) d\sigma(\eta)\right| \geq \int_{0<d(\xi,\eta)<\delta} \frac{\widetilde{C}}{|1-\langle \xi,\eta\rangle|^{2n}} f(\eta) d\sigma(\eta) \tag{2.4}$$

for any nonnegative function f, where $\widetilde{C}$ depends only on the distance between ξ and η. Suppose that Q_1 and Q_2 are nonintersecting with positive distance nonisotropic balls with radius sufficiently small δ, and that they are contained in another small nonisotropic ball, for example, with radius 3δ. Choose a nonnegative function f supported in Q_1. Then from (2.4), for almost all $\xi \in Q_2$ we have

$$|Kf(\xi)| = \left|\int_{Q_1} K(\xi,\eta) f(\eta) d\sigma(\eta)\right| \geq \int_{Q_1} \frac{\widetilde{C}}{|1-\langle \xi,\eta\rangle|^{2n}} f(\eta) d\sigma(\eta) := \widetilde{C} I. \tag{2.5}$$

Since $\sigma(Q_1) \approx \delta^{2n}$, there is a constant $C > 0$ such that $I \geq C(1/\sigma(Q_1) \int_{Q_1} f d\sigma)$. Thus for almost all $\xi \in Q_2$, we get

$$|Kf(\xi)|^p \geq C^p \widetilde{C}^p \left(\frac{1}{\sigma(Q_1)} \int_{Q_1} f d\sigma\right)^p. \tag{2.6}$$

Putting $f = \chi_{Q_1}$ and integrating (2.6) over Q_2 after being multiplied by ω, we get

$$\int_{Q_2} \omega \, d\sigma \leq \frac{1}{C^p \widetilde{C}^p} \int_{Q_2} |Kf(\xi)|^p \omega \, d\sigma. \tag{2.7}$$

However by (1.5) there exists a number C_p such that

$$\int_{Q_2} |Kf|^p \omega \, d\sigma \leq \int_S |Kf|^p \omega \, d\sigma \leq C_p \int_S |f|^p \omega \, d\sigma = C_p \int_{Q_1} \omega \, d\sigma. \tag{2.8}$$

Thus we get

$$\int_{Q_2} \omega \, d\sigma \leq \frac{C_p}{C^p \widetilde{C}^p} \int_{Q_1} \omega \, d\sigma. \tag{2.9}$$

Similarly, putting $f = \chi_{Q_2}$ and integrating (2.6) over Q_1 after being multiplied by ω and then using (1.5), we also have

$$\int_{Q_1} \omega \, d\sigma \leq \frac{C_p}{C^p \widetilde{C}^p} \int_{Q_2} \omega \, d\sigma. \tag{2.10}$$

Therefore, the integrals of ω over Q_1 and Q_2 are equivalent.

Now for a given constant α, put $f = \omega^\alpha \chi_{Q_1}$ in (2.6) and integrate over Q_2. We have

$$\int_{Q_2} |Kf(\xi)|^p \omega \, d\sigma \geq C^p \widetilde{C}^p \left(\frac{1}{\sigma(Q_1)} \int_{Q_1} \omega^\alpha d\sigma \right)^p \int_{Q_2} \omega \, d\sigma. \tag{2.11}$$

Thus we get

$$\left(\frac{1}{\sigma(Q_1)} \int_{Q_1} \omega^\alpha d\sigma \right)^p \int_{Q_2} \omega \, d\sigma \leq \frac{C_p}{C^p \widetilde{C}^p} \int_{Q_1} \omega^{\alpha p + 1} d\sigma. \tag{2.12}$$

Finally take $\alpha = -1/(p-1)$ and apply (2.10) to (2.12), then we have the inequality

$$\frac{1}{\sigma(Q_1)} \int_{Q_1} \omega \, d\sigma \left(\frac{1}{\sigma(Q_1)} \int_{Q_1} \omega^{-1/(p-1)} d\sigma \right)^{p-1} \leq \left(\frac{C_p}{C^p \widetilde{C}^p} \right)^2, \tag{2.13}$$

for every ball Q_1 with radius less than or equal to δ at any point of S. (Here, note that the right hand side of the above is independent of Q_1 and particularly δ because $\widetilde{C}$ depends only on the distance between Q_1 and Q_2.) Therefore,

$$\frac{1}{\sigma(Q)} \int_{Q} \omega \, d\sigma \left(\frac{1}{\sigma(Q)} \int_{Q} \omega^{-1/(p-1)} d\sigma \right)^{p-1} \leq M_p, \tag{2.14}$$

where the constant M_p is independent of Q. Consequently, we have the desired A_p-condition. And this proves the necessity of the A_p-condition for (1.5).

Conversely, we suppose that $1 < p < \infty$ and ω satisfies the A_p-condition and then we will prove that (1.5) holds. To do this we will first prove the following. Claim (i). *Let $f \in L^1(S)$. Then for $q > 1$, there is a constant $C_q > 0$ such that $(Kf)^{\#^1}(\xi) \leq C_q f^{\#^q}(\xi)$, for almost all ξ.*

To prove Claim (i), for a fixed $Q = Q(\xi_Q, \delta)$, it suffices to show that for each $q > 1$ there are constants $\lambda = \lambda(Q, f)$ and C_q depending only on q such that

$$\frac{1}{\sigma(Q)} \int_Q \left| Kf(\eta) - \lambda \right| d\sigma \le C_q f^{\#q}(\xi_Q). \tag{2.15}$$

Now, we write

$$f(\eta) = (f(\eta) - f_Q)\chi_{2Q}(\eta) + (f(\eta) - f_Q)\chi_{S\setminus 2Q}(\eta) + f_Q = f_1(\eta) + f_2(\eta) + f_Q. \tag{2.16}$$

Since $Kf_Q = 0$, we have $Kf = Kf_1 + Kf_2$.
Define

$$g(z) = \int_S (2C(z,\xi) - 1) f_2(\xi) d\sigma(\xi). \tag{2.17}$$

Then g is continuous on $B \cup Q$. By setting $\lambda = -ig(\xi_Q)$ in (2.15), we shall prove the Claim. The integral in (2.15) is estimated as

$$\int_Q \left| Kf(\eta) + ig(\xi_Q) \right| d\sigma(\eta) \le \int_Q \left| Kf_1 \right| d\sigma + \int_Q \left| Kf_2 + ig(\xi_Q) \right| d\sigma = I_1 + I_2. \tag{2.18}$$

Estimate of I_1. By Hölder's inequality we get

$$\begin{aligned} \frac{1}{\sigma(Q)} \int_Q \left| Kf_1 \right| d\sigma &\le \left(\frac{1}{\sigma(Q)} \int_Q \left| Kf_1 \right|^q d\sigma \right)^{1/q} \\ &\le \left(\frac{1}{\sigma(Q)} \int_S \left| Kf_1 \right|^q d\sigma \right)^{1/q} \le \frac{C}{\sigma(Q)^{1/q}} \left\| f_1 \right\|_q, \end{aligned} \tag{2.19}$$

since K is bounded on $L^q(S)$. (Here, throughout the proof for notational simplicity, the letter C alone will denote a positive constant, independent of δ, whose value may change from line to line.) Now by replacing f_1 by $f - f_Q$, we get

$$\left\| f_1 \right\|_q = \left(\int_{2Q} \left| f - f_Q \right|^q d\sigma \right)^{1/q} \le \left(\int_{2Q} \left| f - f_{2Q} \right|^q d\sigma \right)^{1/q} + \sigma(2Q)^{1/q} \left| f_{2Q} - f_Q \right|. \tag{2.20}$$

Thus by applying Hölder's inequality in the last term of the above, we see that there is a constant C_q such that

$$\frac{1}{\sigma(Q)} \int_Q \left| Kf_1 \right| d\sigma \le C_q f^{\#q}(\xi_Q). \tag{2.21}$$

Now we estimate I_2. Since $f_2 \equiv 0$ on $2Q$, we have

$$I_2 = \int_Q \left|f_2 + iKf_2 - g(\xi_Q)\right| d\sigma \leq \int_{S\setminus 2Q} 2\left|f_2(\eta)\right| \int_Q \left|C(\xi,\eta) - C(\xi_Q,\eta)\right| d\sigma(\xi) d\sigma(\eta). \quad (2.22)$$

By Lemma 6.6.1 of [2], we get an upper bound such that

$$I_2 \leq C\delta\sigma(Q) \int_{S\setminus 2Q} \frac{\left|f_2(\eta)\right|}{\left|1 - \langle \eta, \xi_Q\rangle\right|^{n+1/2}} d\sigma(\eta), \quad (2.23)$$

where C is an absolute constant.

Write $S \setminus 2Q = \bigcup_{k=1}^{\infty} 2^{k+1}Q \setminus 2^k Q$. Then the integral of (2.23) is equal to

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} \frac{\left|f(\eta) - f_Q\right|}{\left|1 - \langle \eta, \xi_Q\rangle\right|^{n+1/2}} d\sigma(\eta) \\ &\quad \leq \sum_{k=1}^{\infty} \frac{1}{2^{(2n+1)k}\delta^{2n+1}} \int_{2^{k+1}Q\setminus 2^k Q} \left|f - f_Q\right| d\sigma \\ &\quad \leq \sum_{k=1}^{\infty} \frac{1}{2^{(2n+1)k}\delta^{2n+1}} \left(\int_{2^{k+1}Q} \left|f - f_{2^{k+1}Q}\right| d\sigma + \sum_{j=0}^{k} \int_{2^{k+1}Q} \left|f_{2^{j+1}Q} - f_{2^j Q}\right| d\sigma \right). \end{aligned} \quad (2.24)$$

Thus there exist C and C_q such that

$$\frac{1}{\sigma(Q)} \int_Q \left|Kf_2 + ig(\xi_Q)\right| d\sigma \leq C \sum_{k=1}^{\infty} \frac{k}{2^k} f^{\#^1}(\xi_Q) \leq C_q f^{\#^q}(\xi_Q), \quad (2.25)$$

as we complete the proof of the claim.

Next, we fix $p > 1$ and let $f \in L^p$. Then by Lemma maximal inequality there is a constant C_p such that

$$\int_S |Kf|^p \omega \, d\sigma \leq \int_S |M(Kf)|^p \omega \, d\sigma \leq C_p \int_S \left|(Kf)^{\#^1}\right|^p \omega \, d\sigma. \quad (2.26)$$

Take $q > 0$ such that $p/q > 1$. By the above Claim (i), the last term of the above inequalities is bounded by some constant (depending on p and q) times

$$\int_S \left|f^{\#^q}\right|^p \omega \, d\sigma \leq C \int_S \left(M|f|^q\right)^{p/q} \omega \, d\sigma \leq C' \int_S |f|^p \omega \, d\sigma, \quad (2.27)$$

where two constants C and C' depend on p and q, which proves (1.5) and this completes the proof of Theorem 1.1. □

Now, we will prove Theorem 1.2 by taking slightly a roundabout way from the proof of Theorem 1.1.

Proof of Theorem 1.2. Assume the inequality (1.7). Let Q_1 and Q_2 be nonintersecting non-isotropic balls with positive distance, and with radius sufficiently small δ.

Let f be supported in Q_1. Then from (2.4), there is a positive constant $\widetilde{C}$ such that for all $\xi \in Q_2$,

$$|Kf(\xi)| \geq \widetilde{C} \int_{Q_1} \frac{1}{|1-\langle \xi, \eta \rangle|^{2n}} f(\eta) d\sigma(\eta), \tag{2.28}$$

where $\widetilde{C}$ depends only on the distance between ξ and η. Also from the fact that $\sigma(Q_1) \approx \delta^{2n}$, for some constant $C > 0$ depending only on n, the integral of (2.28) has the lower bound such as

$$C\left(\frac{1}{\sigma(Q_1)} \int_{Q_1} f \, d\sigma\right). \tag{2.29}$$

Thus for almost all $\xi \in Q_2$, we get

$$|Kf(\xi)|^p \geq C^p \widetilde{C}^p \left(\frac{1}{\sigma(Q_1)} \int_{Q_1} f \, d\sigma\right)^p. \tag{2.30}$$

Putting $f = \chi_{Q_1}$ and integrating (2.30) over Q_2 after being multiplied by v, we get

$$\int_{Q_2} v \, d\sigma \leq \frac{1}{C^p \widetilde{C}^p} \int_{Q_2} |Kf(\xi)|^p v \, d\sigma. \tag{2.31}$$

However, by (1.7) there exists a number C_p such that

$$\int_{Q_2} |Kf|^p v \, d\sigma \leq \int_S |Kf|^p v \, d\sigma \leq C_p \int_S |f|^p w \, d\sigma = C_p \int_{Q_1} w \, d\sigma. \tag{2.32}$$

Thus,

$$\int_{Q_2} v \, d\sigma \leq \frac{C_p}{C^p \widetilde{C}^p} \int_{Q_1} w \, d\sigma. \tag{2.33}$$

For a constant α which will be chosen later, put $f = w^{\alpha} \chi_{Q_1}$ in (2.30), multiply v on both sides, and integrate over Q_2. We have

$$\int_{Q_2} |Kf(\xi)|^p v \, d\sigma \geq C^p \widetilde{C}^p \left(\frac{1}{\sigma(Q_1)} \int_{Q_1} w^{\alpha} d\sigma\right)^p \int_{Q_2} v \, d\sigma. \tag{2.34}$$

By (1.7), we arrive at

$$\left(\frac{1}{\sigma(Q_1)}\int_{Q_1} w^{\alpha}\,d\sigma\right)^p \int_{Q_2} v\,d\sigma \leq \frac{C_p}{C^p\widetilde{C}^p}\int_{Q_1} w^{\alpha p+1}\,d\sigma. \tag{2.35}$$

Taking $\alpha = -1/(p-1)$ in (2.35), we have the inequality

$$\frac{1}{\sigma(Q_1)}\int_{Q_2} v\,d\sigma\left(\frac{1}{\sigma(Q_1)}\int_{Q_1} w^{-1/(p-1)}\,d\sigma\right)^{p-1} \leq \left(\frac{C_p}{C^p\widetilde{C}^p}\right)^2, \tag{2.36}$$

for all balls Q_1, Q_2 with radius less than or equal to δ and the distance between two balls greater then δ at any point of S.

Here, unlike the proof of Theorem 1.1, we can not derive the equivalence between $\int_{Q_i} v\,d\sigma$ and $\int_{Q_j} w\,d\sigma$ in a straightforward method, for $i \neq j$ $(i,j = 1,2)$. For this reason, it is not allowed to replace Q_1 by Q_2 directly in (2.36). However, such difficulty can be overcome using the following method. By the symmetric process of the proof, we can interchange Q_1 with Q_2 in (2.36). Thus, for all such balls,

$$\frac{1}{\sigma(Q_2)}\int_{Q_1} v\,d\sigma\left(\frac{1}{\sigma(Q_2)}\int_{Q_2} w^{-1/(p-1)}\,d\sigma\right)^{p-1} \leq \left(\frac{C_p}{C^p\widetilde{C}^p}\right)^2. \tag{2.37}$$

Now multiply two equations (2.36) and (2.37) by side. Since $\sigma(Q_1) = \sigma(Q_2)$, we have

$$\begin{aligned} &\frac{1}{\sigma(Q_1)}\int_{Q_1} v\,d\sigma\left(\frac{1}{\sigma(Q_2)}\int_{Q_2} w^{-1/(p-1)}\,d\sigma\right)^{p-1} \\ &\quad\times\frac{1}{\sigma(Q_2)}\int_{Q_2} v\,d\sigma\left(\frac{1}{\sigma(Q_1)}\int_{Q_1} w^{-1/(p-1)}\,d\sigma\right)^{p-1} \leq \left(\frac{C_p}{C^p\widetilde{C}^p}\right)^4. \end{aligned} \tag{2.38}$$

Let us note that $\widetilde{C}$ depends on the distance between Q_1 and Q_2. Taking supremum over all δ-balls, we get

$$\left(\sup_Q \frac{1}{\sigma(Q)}\int_Q v\,d\sigma\left(\frac{1}{\sigma(Q)}\int_Q w^{-1/(p-1)}\,d\sigma\right)^{p-1}\right)^2 \leq \left(\frac{C_p}{C^p\widetilde{C}^p}\right)^4, \tag{2.39}$$

and the proof of Theorem 1.2 is complete. □

Appendix

A_p-Condition and BMO

Let Q be a nonisotropic ball of S. The space BMO consists of all $f \in L^1(S)$ satisfying

$$\sup_Q \frac{1}{\sigma(Q)} \int_Q |f - f_Q| d\sigma = \|f\|_{BMO} < \infty, \tag{A.1}$$

where f_Q is the average of f over Q. BMO becomes a Banach space provided that we identify functions which differ by a constant. Since both definitions of A_p-condition and BMO are concerned about the local average of a function, it is natural for us to mention the relation between these concepts. In this section, we show that an A_p weight on S is indeed closely related to the BMO. Proposition A.4 and Lemma A.3 tell about it. The proof of Proposition A.4 comes from John-Nirenberg's inequality (Lemma A.3) which states as follows.

Lemma A.3 (John-Nirenberg's inequality). *Let $f \in BMO$ and $E \subset S$ be not intersecting the north pole. Then there exist positive constants C_1 and C_2, independent of f and E, such that*

$$\sigma(\{\eta \in E : |f(\eta) - f_E| > \lambda\}) \leq C_1 e^{-C_2\lambda/\|f\|_{BMO}} \sigma(E) \tag{A.2}$$

for every $\lambda > 0$.

The proof of Lemma A.3 is parallel to the proof of the classical John-Nirenberg's inequality on $\mathbb{R}$ [3, Theorem 2.1 of Chapter 6]. However, it is somewhat more complicated, and moreover, the details of the proof run off our aim of the paper. So we decide to omit the proof of Lemma A.3.

The next proposition is about the A_p weight and BMO on S. Likewise, on the Euclidean space, by Jensen's inequality and the classical John-Nirenberg's inequality, we can see that the Euclidean analogue of Proposition A.4 is also true.

Proposition A.4. *Let ω be a nonnegative integrable function on S. Then $\log \omega \in BMO$ if and only if ω^α satisfies the A_2-condition for some $\alpha > 0$.*

Proof. We prove the necessity first. Suppose $\log \omega \in BMO$. Let Q denote a nonisotropic ball, and $\alpha > 0$. Now consider integral

$$\frac{1}{\sigma(Q)} \int_Q e^{\alpha|\log \omega - (\log \omega)_Q|} d\sigma, \tag{A.3}$$

which is less than or equal to

$$1 + \frac{1}{\sigma(Q)} \int_1^\infty \sigma\left(\left\{\eta \in Q : e^{\alpha|\log \omega(\eta) - (\log \omega)_Q|} > \lambda\right\}\right) d\lambda. \tag{A.4}$$

By change of variables, the integral term of the above is equal to

$$\frac{\alpha}{\sigma(Q)}\int_0^\infty \sigma\left(\left\{\eta\in Q:\left|\log\omega(\eta)-(\log\omega)_Q\right|>\lambda\right\}\right)e^{\alpha\lambda}d\lambda. \tag{A.5}$$

John-Nirenberg's inequality implies that there exist positive constants C_1 and C_2, independent of Q, such that

$$\sigma\left(\left\{\eta\in Q:\left|\log\omega(\eta)-(\log\omega)_Q\right|>\lambda\right\}\right)\le C_1e^{-C_2\lambda/\|\log\omega\|_{BMO}}\sigma(Q). \tag{A.6}$$

Now we take $\alpha < C_2/\|\log\omega\|_{BMO}$, and then we define

$$M=\frac{C_1C_2}{C_2-\alpha\|\log\omega\|_{BMO}}. \tag{A.7}$$

By the above choice of α and M, for each nonisotropic ball Q, we have the inequality

$$\frac{1}{\sigma(Q)}\int_Q e^{\pm\alpha(\log\omega-(\log\omega)_Q)}d\sigma\le M+1. \tag{A.8}$$

Therefore we have

$$\sup_Q\frac{1}{\sigma(Q)}\int_Q e^{\alpha\log\omega}d\sigma\left(\frac{1}{\sigma(Q)}\int_Q e^{-\alpha\log\omega}d\sigma\right)\le(M+1)^2, \tag{A.9}$$

which means that ω^α satisfies the A_2-condition.

Conversely, suppose that there is $\alpha>0$ such that ω^α satisfies the A_2-condition. Then by Jensen's inequality it suffices to show that

$$\sup_Q\frac{1}{\sigma(Q)}\int_Q e^{\alpha|\log\omega-(\log\omega)_Q|}d\sigma<\infty. \tag{A.10}$$

Let us note that

$$\begin{aligned}\frac{1}{\sigma(Q)}\int_Q e^{\alpha|\log\omega-(\log\omega)_Q|}d\sigma &\le \frac{1}{\sigma(Q)}\int_Q e^{\alpha\log\omega}d\sigma\, e^{-\alpha(\log\omega)_Q}+\frac{1}{\sigma(Q)}\int_Q e^{-\alpha\log\omega}d\sigma\, e^{\alpha(\log\omega)_Q}\\ &= I+II.\end{aligned} \tag{A.11}$$

Since both integrals I and II are bounded in essentially the same way, we only do I. From Jensen's inequality once more, we have

$$I=\left(\frac{1}{\sigma(Q)}\int_Q e^{\alpha\log\omega}d\sigma\right)e^{\sigma(Q)^{-1}\int_Q\log\omega^{-\alpha}d\sigma}\le\left(\frac{1}{\sigma(Q)}\int_Q\omega^{\alpha}d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q\omega^{-\alpha}d\sigma\right). \tag{A.12}$$

Since ω^{α} satisfies the A_2-condition, we finish the sufficiency and this completes the proof of the proposition. □

Let ω satisfy the A_p-condition and $r>p$. Then, since $1/(r-1)<1/(p-1)$, Hölder's inequality implies that

$$\left(\frac{1}{\sigma(Q)}\int_Q\omega^{-1/(r-1)}d\sigma\right)^{1/(r-1)}\le\left(\frac{1}{\sigma(Q)}\int_Q\omega^{-1/(p-1)}d\sigma\right)^{1/(p-1)}. \tag{A.13}$$

This means that ω satisfies the A_r-condition. Also we can easily see that $\omega^{-1/(p-1)}$ satisfies the A_q-condition for $q=p/(p-1)$. From this and Proposition A.4, we get the following corollary.

Corollary A.5. *Let $p>1$ and let ω be a nonnegative integrable function on S such that ω^{α} satisfies the A_p-condition for some $\alpha>0$. Then $\log\omega\in BMO$.*

Proof. If $p\le 2$, then ω^{α} satisfies the A_2-condition. Thus Proposition A.4 implies $\log\omega\in BMO$. If $p>2$, then $\omega^{-\alpha/(p-1)}$ satisfies the A_q-condition for $q=p/(p-1)<2$, which implies that $\omega^{-\alpha/(p-1)}$ satisfies the A_2-condition. Thus by Proposition A.4, we get $\log\omega^{-\alpha/(p-1)}\in BMO$, consequently $\log\omega\in BMO$. □

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